

Reducibility of joint relay positioning and flow optimization problem

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Abstract—This paper shows how to reduce the otherwise hard joint relay positioning and flow optimization problem into a sequence of two simpler decoupled problems. We consider a class of wireless multicast hypergraphs mainly characterized by their hyperarc rate functions, that are increasing and convex in power, and decreasing in distance between the transmit node and the farthest end node of the hyperarc. The set-up consists of a single multicast flow session involving a source, multiple destinations and a relay that can be positioned freely. The first problem formulates the relay positioning problem in a purely geometric sense, and once the optimal relay position is obtained the second problem addresses the flow optimization. Furthermore, we present simple and efficient algorithms to solve these problems.

I. INTRODUCTION

We consider a version of network planning problem under a relatively simple construct of a single session consisting of a source s , a destination set T and an arbitrarily positionable relay r , all on a 2-D Euclidean plane. The problem can then be stated as: *What is optimal relay position that maximizes the multicast flow from s to T ?* Similarly, we can also ask: *What is the optimal relay position that minimizes the cost (in terms of total network power) for a target multicast flow F ?*

A fairly general class of acyclic hypergraphs are considered. The hypergraph model is characterized by the following rules of construction of the hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$:

- 1) $\mathcal{G}(\mathcal{N}, \mathcal{A})$ consists of finite set of nodes \mathcal{N} positioned on a 2-D Euclidean plane and a finite set of hyperarcs \mathcal{A} .
- 2) Each hyperarc in \mathcal{A} emanates from a transmit node and connects a set of receivers (or end nodes) in the system. Also, each hyperarc is associated with a rate function that is convex and increasing in transmit node power and decreasing in distance between the transmit node and the farthest node spanned by the hyperarc in the system.
- 3) Each end node spanned by the hyperarc can decode the information sent over the hyperarc equally reliably, i.e. all the end nodes of an hyperarc get equal rate.

In relation to the special case of our hypergraph model, the authors addressed the first question (max-flow) in the context of Low-SNR Broadcast Relay Channel in [1].

This paper has two major contributions. Firstly, we solve the general joint relay positioning and max-flow optimization problem for our hypergraph model. Secondly, we address the

min-cost flow problem and establish a relation of duality between the max-flow and min-cost problems. An efficient algorithm that solves the joint relay positioning and max-flow problem is presented, in addition to an algorithm that solves an important special case of the min-cost problem.

The relay positioning problem has been studied in various settings [2]–[4]. In most cases, the problem is either heuristically solved due to inherent complexity, or approximately solved using simpler methods but compromising accuracy. We reduce the non-convex joint problem into easily solvable sequence of two decoupled problems. The first problem solves for optimal relay position in a purely geometric sense with no flow optimization involved. Upon obtaining the optimal relay position, the second problem addresses the flow optimization. The decoupling of the joint problem comes as a consequence of the convexity (in power) of hyperarc rate functions.

The next section develops the wireless network model. Section III presents the key multicast flow concentration ideas for max-flow and min-cost flow that are central to the reducibility of the joint problem. In Section IV, we present the algorithms and Section V contains an example where the results of this paper are applied. Finally, we conclude in Section VI.

II. PRELIMINARIES AND MODEL

Consider a wireless network hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$ consisting of $|\mathcal{N}| = n+2$ nodes placed on a 2-D Euclidean plane with $|\mathcal{A}|$ number of hyperarcs and the only arbitrarily positionable node as the relay r . The node set $\mathcal{N} = \{s, r, t_1, \dots, t_n\}$ consists of a source node s , a relay r and an ordered destination set $T = \{t_1, \dots, t_n\}$ (in increasing distance from s). Their positions on the 2-D Euclidean plane are denoted by the set of two-tuple vector $\mathcal{Z} = \{z_i = (x_j, y_j) | \forall j \in \mathcal{N}\}$.

All hyperarcs in \mathcal{A} are denoted by (u, V_{k_u}) , where u is the transmit node and $V_{k_u} = \{v_1, \dots, v_{k_u}\}$ is the ordered set (in increasing distance u) of end nodes of the hyperarc, and $V_{k_u} \subset \mathcal{N} \setminus \{u\}$. The hyperarcs emanating from a transmitter node are constructed in order of increasing distances of the receivers from the transmitter (refer Figure 1). This construction rule captures the distance based approach and is analogous to time sharing for broadcasting. Note that, this is one technique to construct the hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$, our model allows arbitrary styles of hypergraph construction that follow

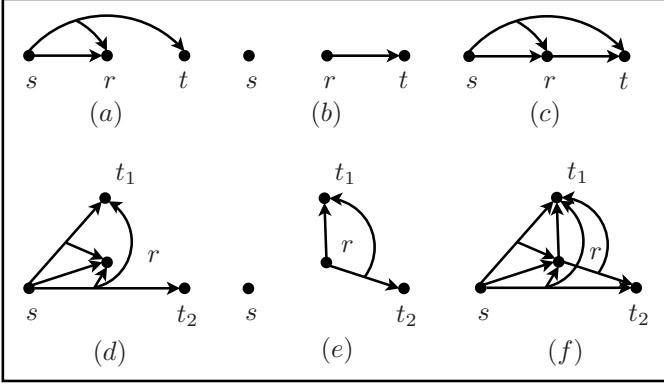


Fig. 1. Hyperarcs are constructed in increasing order of distance from the transmitter. (a)-(c): 3 node system. (a): Source hyperarc set - $\{(s, r), (s, rt)\}$. (b): Relay hyperarc set - $\{(r, t)\}$. (c): Hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$. (d)-(f): 4 node system with $T = \{t_1, t_2\}$ such that $D_{sr} < D_{st_1} < D_{st_2}$ and $D_{rt_1} < D_{rt_2}$. (d) Source hyperarc set - $\{(s, r), (s, rt_1), (s, rt_1t_2)\}$. (e) Relay hyperarc set - $\{(r, t_1), (r, t_1t_2)\}$. (f): Hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$.

the above three mentioned rules. Although, since time sharing is optimal for broadcasting we will stick to this technique as the main example in this paper. All the nodes in the set V_{k_u} receive the information transmitted over the hyperarc (u, V_{k_u}) equally reliably. Any hyperarc $(u, V_{k_u}) \in \mathcal{A}$ is associated with a rate function $R_{v_{k_u}}^u = f(P_{v_{k_u}}^u, D_{uv_{k_u}})$, where $P_{v_{k_u}}^u$ and $D_{uv_{k_u}}$ denotes the fraction of the total transmit node power allocated for the hyperarc and the Euclidean distance between transmit node u and the farthest end node v_{k_u} , respectively.

The hyperarc rate function $R_{v_{k_u}}^u$ is increasing and convex in power $P_{v_{k_u}}^u$ and decreasing in $D_{uv_{k_u}}$. Furthermore, without loss of generality, we write the hyperarc rate function into two separable functions of power and distance

$$R_{v_{k_u}}^u = \frac{g(P_{v_{k_u}}^u)}{h(D_{uv_{k_u}})} \text{ or } R_{v_{k_u}}^u = g(P_{v_{k_u}}^u) - h(D_{uv_{k_u}}), \quad (1)$$

where $g : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is increasing and convex and $h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is increasing. Mainly, we will be concerned with the first equation in (1). Moreover, to comply with standard physical wireless channel models we assume that

$$\frac{\partial g(P_{v_{k_u}}^u)}{\partial P_{v_{k_u}}^u} \leq \frac{\partial h(D_{uv_{k_u}})}{\partial D_{uv_{k_u}}}, \quad (2)$$

$\forall (P_{v_{k_u}}^u = D_{uv_{k_u}}) \in \text{dom}(P_{v_{k_u}}^u, D_{uv_{k_u}})$. If the functions g and h are not differentiable entirely in $\text{dom}(P_{v_{k_u}}^u, D_{uv_{k_u}})$, then Inequality 2 can be rewritten with partial sub-derivatives, implying that differentiability is not imperative.

Denote the convex hull of the nodes in $\mathcal{N} \setminus \{r\}$ by \mathcal{C} . For a given relay position $z_r \in \mathcal{C}$, let $L_i = \{l_1^i, \dots, l_{\tau_i}^i\}$ be the set of paths from s to a destination $t_i \in T$ and let $L = \{l_1, \dots, l_{\tau}\}$ be the set of paths from s that span all the destination set T , therefore $L \subset \bigcup_{i \in [1, n]} L_i$. Moreover, any path in the system consists of either a single hyperarc or at most two hyperarcs as there are only two transmitters in the system. Let μ and ν denote the total given power of source and relay, respectively, and $\gamma = \frac{\nu}{\mu}$ denote their ratio, where $\gamma \in (0, \infty)$. Denote with $F_{l_j^i}$ and \bar{F}_i the flow over the path l_j^i (for $j \in [1, \tau_i]$) and the total flow to the destination $t_i \in T$, respectively, such that

$F_i = \sum_{j \in [1, \tau_i]} F_{l_j^i}$. Define F to be the the multicast flow from s to the destination set T as the minimum among the total flows to each destination, then for a given relay position $z_r \in \mathcal{C}$ the multicast max-flow problem can be written as,

$$\text{Maximize} \quad \left(F = \min_{i \in [1, n]} F_i \right) \quad (A)$$

$$\text{subject to: } F_i \leq \sum_{j=1}^{\tau_i} F_{l_j^i}, \forall i \in [1, n], \quad (3)$$

$$0 \leq F_{l_j^i} \in \mathfrak{C}(P, D), \quad \forall j \in [1, \tau_i], \forall i \in [1, n]. \quad (4)$$

The hyperarc rate constraints and node sum-power constraints are denoted by the set $\mathfrak{C}(P, D)$ in Program (A) for simplicity. Program (A) in general is non-convex, as the path flow function $F_{l_j^i}$ can be non-convex, e.g. let the path $l_j^i \in L_i$ be $l_j^i = \{(s, V_{k_s}), (r, V_{k_r})\}$, $(l_1^{t_2} = \{(s, rt_1), (r, t_1t_2)\})$ in Figure 1(f), then $F_{l_j^i} = \min(R_{v_{k_s}}^s, R_{v_{k_r}}^r)$.

Now we define the notion of cost for a given hyperarc rate $R_{v_{k_u}}^u = \frac{g(P_{v_{k_u}}^u)}{h(D_{uv_{k_u}})} \geq 0$. The cost of rate $R_{v_{k_u}}^u$ is given by the total power consumed by the hyperarc to achieve $R_{v_{k_u}}^u$

$$P_{v_{k_u}}^u = g^{-1} \left(R_{v_{k_u}}^u h(D_{uv_{k_u}}) \right), \quad (5)$$

where $g^{-1} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is the inverse function of g that maps its range to its domain. Therefore, the total cost of multicast flow F is simply the sum of powers of all the hyperarcs in the system. Note that the function g^{-1} is increasing and concave, and if h is convex then from Inequality (2), $g^{-1} \circ h$ increasing and convex. So for a given relay position $z_r \in \mathcal{C}$, the min-cost problem minimizing the total cost for setting up the multicast session (s, T) with a target flow F can be written as,

$$\text{Minimize} \quad \left(P = \sum_{(u, V_{k_u}) \in \mathcal{A}} P_{v_{k_u}}^u \right) \quad (B)$$

$$\text{subject to: } F \leq F_i \leq \sum_{j=1}^{\tau_i} F_{l_j^i}, \quad \forall i \in [1, n], \quad (6)$$

$$\mathfrak{C}(P, D) \ni F_{l_j^i} \geq 0, \forall j \in [1, \tau_i], \forall i \in [1, n]. \quad (7)$$

Constraint (6) makes sure that any destination $t_i \in T$ receives a minimum of flow F . Like in Program (A), we denote with the set $\mathfrak{C}(P, D)$ the hyperarc rate and power constraints.

Finally, define the point p^* , that will be crucial in developing algorithms in later sections, as

$$z_{p^*} = \arg \min_{z_p} (\max(\nu^* h(D_{z_p s}), \mu^* \max_{t_i \in T} (h(D_{z_p t_i})))), \quad (8)$$

where, $\mu^* = g(\mu)$ and $\nu^* = g(\nu)$. An easy way to understand p^* is that if $\mu^* = \nu^* = 1$ then p^* is the circumcenter of two or more nodes in the set $\mathcal{N} \setminus \{r\}$. Note that the program in Equation (8) is a convex program. Also, denote the optimal value of the objective function in Equation (8) as D_{p^*} .

Hereafter, we represent with $(s, T, \mathcal{Z}, \gamma)$ and $(s, T, \mathcal{Z}, \gamma, F)$ the joint relay positioning and flow optimization problem instances that maximizes the multicast flow and minimizes the total cost for a the target flow F , and with $z_{\gamma \uparrow}^*$ and $z_{F \downarrow}^*$ denote the optimal relay positions, respectively.

III. MULTICAST FLOW PROPERTIES AND REDUCTION

In this section we develop fundamental multicast flow properties that govern the multicast flow in the wireless network hypergraphs that we consider in this paper. First, we briefly note the main hurdles in jointly optimizing the problem. For a given problem instance different relay positions can result in different hypergraphs, which makes the use of standard graph-based flow optimization algorithms difficult. Moreover, the hyperarc rate function can be non-convex itself.

We will show that the joint problems $(s, T, \mathcal{Z}, \gamma)$ and $(s, T, \mathcal{Z}, \gamma, F)$ can be reduced to solving a sequence of two decoupled problems. The reduced problems are decoupled in the sense that the first problem is purely a geometric optimization problem and involves no flow optimization and vice versa for the second problem. At the same time, they are not entirely decoupled because the two problems need to be solved in succession and cannot be solved separately. Now we present a series of results that are fundamental to the reducibility of the joint problem.

Proposition 1: The optimal relay positions $z_{\gamma\uparrow}^*$ and $z_{F\downarrow}^*$ lie inside the convex hull \mathcal{C} .

Refer Appendix A for the proof. Proposition 1 tells us that only the points inside the polygon \mathcal{C} need to be considered. This brings us to the following fundamental theorem.

Theorem 1 (Flow Concentration): Given $z_r \in \mathcal{C}$:

- (i) the maximized multicast flow F^* concentrates over at most two paths from s to the destination set T .
- (ii) for any target flow $F \in [0, F^*]$ the min-cost multicast flow concentrates over at most two paths from s to T .

The proof is detailed in Appendix B. Theorem 1 is central to the two questions we aim to answer and reduces the complexity of joint optimization greatly by considering only two paths instead of many. Essentially, Theorem 1 tells that for a given relay position $z_r \in \mathcal{C}$, the multicast flow F must go only over the paths that span all the destination set T , i.e. set L . Furthermore, among the paths in L , the maximized multicast flow F^* goes over only two paths, namely the path $\hat{l}_1 = \{(s, T_1), (r, T_2)\}$ that has the highest min-cut among all the paths through the relay r , and path $\hat{l}_2 = \{(s, t_1, \dots, t_n) = (s, T)\}$, which is the biggest hyperarc from s spanning all the destination set T , where $r \in T_1$ and $T_1 \cup T_2 = T$. The same holds for the min-cost case for a given relay position $z_r \in \mathcal{C}$. Consequently, it is also true for the optimal relay positions $z_{\gamma\uparrow}^*$ and $z_{F\downarrow}^*$. Hereafter, we only need to consider the flow over paths \hat{l}_1 and \hat{l}_2 (corresponding to the relay position in consideration).

A. Max-flow Problem - $(s, T, \mathcal{Z}, \gamma)$

Assuming that the transmitted signal propagates omnidirectionally, we can geometrically represent the hyperarcs of the path $\hat{l}_1 = \{(s, T_1), (r, T_2)\}$ by circles $C_{T_1}^s$ and $C_{T_2}^r$ centered at s and r with radii $\pi_s = D_{st_k}$ and $\pi_r = D_{rt_{k'}}$ (where $D_{st_k} = \max_{t_i \in T_1} (D_{st_i})$ and $D_{rt_{k'}} = \max_{t_j \in T_2} (D_{rt_j})$), respectively. Similarly, the path $\hat{l}_2 = \{(s, T)\}$ can be represented by the circle C_T^s with radius D_{st_n} . Also, $\mathcal{C}_U = C_{T_1}^s \cup C_{T_2}^r$ denotes

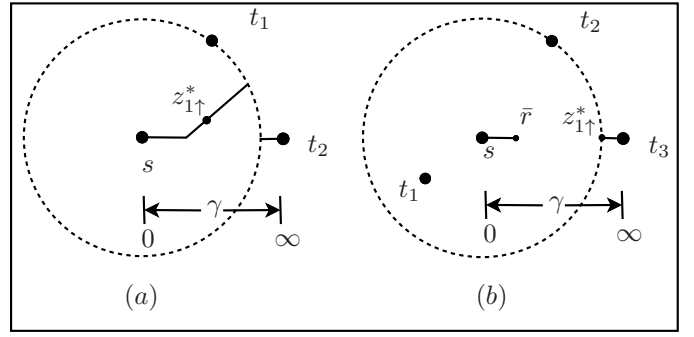


Fig. 2. The solid piecewise linear segment in examples (a) and (b) marks the set of points \hat{r} for different values of $\pi_s \in (0, D_{st_2})$. Each point \hat{r} corresponds to $z_{\gamma\uparrow}^*$ for some $\gamma \in (0, \infty)$. The piecewise linear segment breaks beyond the dashed circle as $z_1 \in C_{T_1}^s$. (a): E.g. C_r^s with $0 < \pi_s < D_{st_1}$, $z_{\hat{r}} = \arg \min_{z_r \in C_r^s} \max(D_{z_r t_1}, D_{z_r t_2})$. Same goes for the example in (b).

the union region of the two circles. Then using Theorem 1, Program (A) can be re-written as,

$$\text{Maximize}_{\substack{P_{T_1}^s + P_{T_2}^r \leq \mu, \\ P_{T_2}^r \leq \nu, \pi_s, \pi_r}} \left(\min \left(\frac{g(P_{T_1}^s)}{h(\pi_s)}, \frac{g(P_{T_2}^r)}{h(\pi_r)} \right) + \frac{g(P_T^s)}{h(D_{st_n})} \right) \quad (C)$$

where, $P_{T_1}^s$, $P_{T_2}^r$ and P_T^s are the powers for hyperarcs of the paths $\hat{l}_1 = \{C_{T_1}^s, C_{T_2}^r\}$ and $\hat{l}_2 = \{C_T^s\}$, respectively. The radii variables π_s and π_r correspond to path \hat{l}_1 for the relay position $z_r \in \mathcal{C}$ such that $z_r \in C_{T_1}^s$ and $\mathcal{Z} \in \mathcal{C}_U$.

Although Program (C) is reduced, it is still a non-convex optimization problem. The objective function is non-convex and different positions of the relay $z_r \in \mathcal{C}$ result in different end node sets T_1 and T_2 for the hyperarcs of path \hat{l}_1 .

On the other hand, we know that the relay position is sensitive only to the flow over path \hat{l}_1 . In addition, as there always exist a relay position $z_r \in \mathcal{C}$ such that the min-cut of path \hat{l}_1 is higher than that of path \hat{l}_2 , then this also holds true for $z_{\gamma\uparrow}^*$. Therefore, optimizing the relay position to maximize the flow over path \hat{l}_1 results in global optimal relay position solving the original problem $(s, T, \mathcal{Z}, \gamma)$. This motivates the decoupling of computation of optimal relay position from the flow maximization over the path \hat{l}_1 .

Proposition 2: For a given problem instance $(s, T, \mathcal{Z}, \gamma)$, if $g(\nu)h(D_{sp^*}) = D_{p^*}$, then $z_{\gamma\uparrow}^* = z_{p^*}$.

Refer Appendix C for the detailed proof. At point p^* , in general the following holds $\frac{g(\mu)}{h(\pi_s^{p^*})} \geq \frac{g(\nu)}{h(\pi_r^{p^*})}$ (from Equation (8)), thus making it naturally a good candidate for $z_{\gamma\uparrow}^*$. Proposition 2, essentially proves that if the relay is positioned at p^* and we get $\frac{g(\mu)}{h(\pi_s^{p^*})} = \frac{g(\nu)}{h(\pi_r^{p^*})}$, and if maximizing the flow over the path \hat{l}_1 results in no spare source power (i.e. $g(\nu)h(D_{sp^*}) = D_{p^*}$), then $z_{\gamma\uparrow}^* = z_{p^*}$ and $F^* = \frac{g(\mu)}{h(\pi_s^{p^*})}$. Furthermore, the joint problem in Program (C) can be reduced to solving in sequence the computation of the optimal relay position p^* by solving Equation (8) and then calculating the max-flow F^* . But this is not true when $\frac{g(\mu)}{h(\pi_s^{p^*})} > \frac{g(\nu)}{h(\pi_r^{p^*})}$. We cover this case in the section of algorithms.

Let us now see the problem in a different way. Consider the radius $\pi_s \in (0, D_{st_n})$ and construct the hyperarc $C_{\pi_s}^s$. Denote

by $T' = \{t_j \in T | D_{st_j} > \pi_s\}$, the set of destination nodes that lie outside the hyperarc circle $C_{\pi_s}^s$. Then compute the point \hat{r} such that

$$z_{\hat{r}} = \arg \min_{z_p \in C_{\pi_s}^s} (\max_{t_j \in T'} (D_{r't_j})),$$

and position the relay at \hat{r} (here \hat{r} is the point in $C_{\pi_s}^s$ such that the maximum among the distances to the nodes in the set T' from \hat{r} is minimized). If $D_{s\hat{r}} < \pi_s$, then we contract the hyperarc $C_{\pi_s}^s$ to $C_{\hat{r}}^s$, else we simply re-denote it with $C_{\hat{r}}^s$. Finally, we can construct the hyperarc $C_{\hat{r}}^s$ (note that $\mathcal{Z} \in C_{\hat{r}}^s \cup C_{\hat{r}}^s$). The set \mathcal{R}' of points \hat{r} computed in this way for different values of $\pi_s \in (0, D_{st_n})$ are the optimal relay positions $z_{\gamma\uparrow}^*$ solving $(s, T, \mathcal{Z}, \gamma)$ for some $\gamma \in (0, \infty)$. Figure 2(a) captures this interesting insight of the relationship between the points \hat{r} and $z_{\gamma\uparrow}^*$. Note that the set $\hat{\mathcal{R}}$ of points \hat{r} is a discontinuous piecewise linear segment.

B. Min-cost Problem $(s, T, \mathcal{Z}, \gamma, F)$ And Duality

The min-cost problem $(s, T, \mathcal{Z}, \gamma, F)$ can be written as

$$\text{Minimize } (P_{T_1}^s + P_{T_2}^r + P_T^s) \quad (\text{D})$$

$$\text{subject to: } F \leq \min \left(\frac{g(P_{T_1}^s)}{h(\pi_s)}, \frac{g(P_{T_2}^r)}{h(\pi_r)} \right) + \frac{g(P_T^s)}{h(D_{st_n})}, \quad (9)$$

$$P_{T_1}^s + P_T^s \leq \mu, \quad P_{T_2}^r \leq \nu. \quad (10)$$

In the non-convex Program (D), the path $\hat{l}_1 = \{C_{T_1}^s, C_{T_2}^r\}$ correspond to the relay position $z_r \in \mathcal{C}$ which is implicitly represented in the distance variables π_s and π_r . From Theorem 1, we know that paths \hat{l}_1 and \hat{l}_2 carry all the min-cost target multicast flow F . In this sub-section we refer the path \hat{l}_1 as the cheapest path for a unit flow among all the paths through r in \mathcal{L} for given position of relay.

Now, we claim that $z_{F\downarrow}^* \in \hat{\mathcal{R}}$. This is true because given the hyperarc $C_{T_1}^s$ of path \hat{l}_1 with optimal radius π_s^* , the second hyperarc $C_{T_2}^r$ must be centered at the point that minimizes the maximum among the distances to all the destination nodes not spanned by the hyperarc $C_{T_1}^s$ from itself, as this minimizes the cost over the hyperarc $C_{T_2}^r$. Therefore, $z_{F\downarrow}^*$ (like $z_{\gamma\uparrow}^*$) always lie on on the curve $\hat{\mathcal{R}}$. This observation motivates an interesting fundamental relationship between $z_{F\downarrow}^*$ and $z_{\gamma\uparrow}^*$.

Theorem 2 (Max-flow/Min-cost Duality): For $F \in [0, F^*]$,

$$z_{F\downarrow}^* = z_{\hat{\gamma}\uparrow}^*, \quad (11)$$

where $\hat{\gamma} \in [\min(\bar{\gamma}, \gamma), \max(\bar{\gamma}, \gamma)]$ and $z_{1\downarrow}^* = z_{\bar{\gamma}\uparrow}^*$.

Theorem 2 establishes the underlying duality relation between the max-flow problem $(s, T, \mathcal{Z}, \gamma)$ and the min-cost problem $(s, T, \mathcal{Z}, \gamma, F)$ and says that the point $z_{F\downarrow}^*$ (or $z_{\hat{\gamma}\uparrow}^*$) lies on the segment $z_{1\downarrow}^* - z_{F^*\downarrow}^* (z_{\hat{\gamma}\uparrow}^* - z_{\gamma\uparrow}^*)$, respectively) of the curve $\hat{\mathcal{R}}$. Implying that the optimal relay position $z_{F\downarrow}^*$ solving the problem $(s, T, \mathcal{Z}, \gamma, F)$ is also the optimal relay position $z_{\hat{\gamma}\uparrow}^*$ solving the problem $(s, T, \mathcal{Z}, \gamma)$ for some $\hat{\gamma}$. The proof of Theorem 2 is presented in Appendix D.

However, the max-flow is not always reducible to a sequence of decoupled problems. This is mainly due to the fact

that the path \hat{l}_2 can be cheaper than path \hat{l}_1 for a unit flow corresponding to the optimal position $z_{F\downarrow}^*$, i.e.

$$g^{-1}(h(\pi_s^*)) + g^{-1}(h(\pi_r^*)) \geq g^{-1}(h(D_{st_n})).$$

This information is not easy to get a priori. In contrast, we can safely assume that

$$g^{-1}(h(\pi_s^*)) + g^{-1}(h(\pi_r^*)) \leq g^{-1}(h(D_{st_n})), \quad (12)$$

as almost all wireless network models that comply with our model result in the hyperarc cost function $g^{-1}(h(D_{uvk_u}))$ being the increasing convex function of distance D_{uvk_u} that satisfy Inequality (12). If Inequality (12) holds, then similar to the Max-flow problem the joint optimal relay positioning and min-cost flow optimization problem in Program (D) can be reduced to a sequence of decoupled problems of computing the optimal relay position and then optimizing the hyperarc powers to achieve the min-cost flow F in the network using the similar arguments as in previous subsection. For a special of the min-cost problem $(s, T, \mathcal{Z}, \gamma, F)$, we present the Min-cost Algorithm that sequentially solves and outputs the optimal relay position and powers to achieve the target flow $F \in [0, F^*]$ in Section IV-B.

IV. ALGORITHMS

In this section we present the general max-flow and the special case min-cost algorithms that solve the sequence of decoupled problems.

A. Max-flow Algorithm

Input: Problem instance $(s, T, \mathcal{Z}, \gamma)$.

- 1: Compute p^* , if $g(\nu)h(D_{sp^*}) = g(\mu)h(D_{p^*t_n})$, output $z_{\gamma\uparrow}^* = z_{p^*}$, $F^* = g(\nu)h(D_{sp^*})$ and quit, else go to 2.
- 2: Construct the set $T' = \{t'_j \in T | D_{st'_j} < D_{p^*t'_j}\} = \{t'_1, \dots, t'_{j'}\}$ (ordered in increasing distance from s) and compute $p_{T' \setminus T'}^*$. If $D_{st'_{j'}} \leq D_{sp_{T' \setminus T'}^*}$, declare $z_{\gamma\uparrow}^* = z_{p_{T' \setminus T'}^*}$ and $F^* = g(\nu)h(D_{sp_{T' \setminus T'}^*})$ and quit, else go to Step 3.
- 3: Compute the points z_1^* and z_2^* , and maximized multicast flow F_1^* and F_2^* , respectively. Declare before quitting,

$$z_{\gamma\uparrow}^* = \begin{cases} z_1^* & \text{if } F_1^* > F_2^*, \\ z_2^* & \text{if } F_1^* < F_2^*. \end{cases}$$

Output: $z_{\gamma\uparrow}^*$ and F^* .

Fig. 3. Max-flow Algorithm.

The Max-flow Algorithm in Figure 3, is a simple and non-iterative 3 step algorithm that outputs the optimal relay position and the maximized multicast flow. The first step is essentially Proposition 2, in case it is not satisfied the second step filters the redundant nodes that are too close to the source and can be ignored. If the conditions of first or second step are not met, then the third step divides the computation of $z_{\gamma\uparrow}^*$ into two regions of \mathcal{C} and computes the optimal relay position z_1^* and z_2^* for these two regions and outputs the better one. The proof of optimality is provided in Appendix E.

B. Min-cost Algorithm

In this subsection, we assume that the Inequality (12) is satisfied and the target flow $F \in [0, F^*]$ goes over the path \hat{l}_1 (corresponding to the optimal relay position $z_{F\downarrow}^*$) only. Min-cost Algorithm in Figure 4, unlike the Max-flow algorithm, is an iterative algorithm. In the first step the geometric feasibility region is constructed and in the second step this region is divided into at most $n - 1$ sub-regions. The optimal relay position is computed for all the sub-regions and the one minimizing the cost among them is declared global optimal. Computing the optimal relay position for the sub-regions is a simple geometric convex program that can be solved efficiently and the number of such iterations is upper bounded by $n - 1$. The proof of optimality is presented in Appendix F.

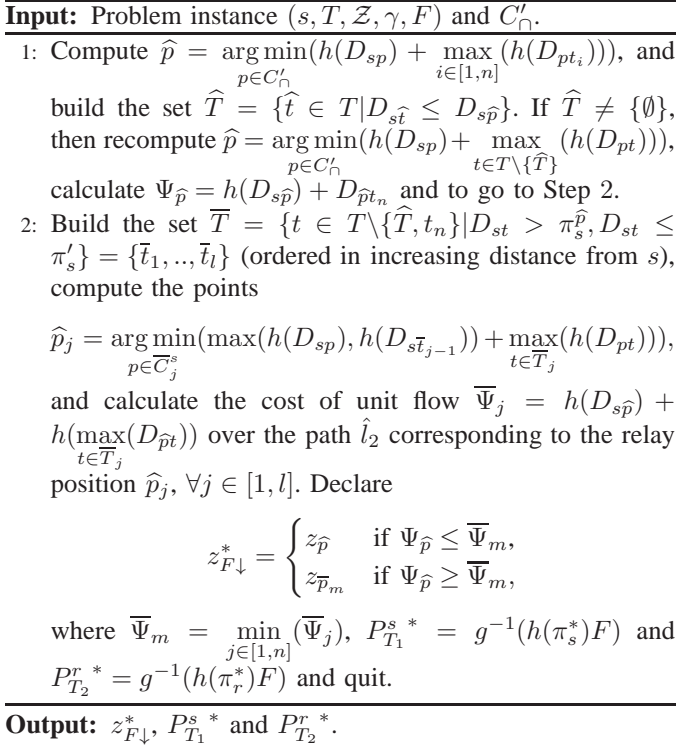


Fig. 4. Min-Cost Algorithm.

V. EXAMPLE: LOW-SNR ACHIEVABLE NETWORK MODEL

In this section we present an example from the interference delimited network model that was originally presented in [1].

A. Low-SNR Broadcast and MAC Channel Model

Consider the AWGN Low-SNR (wideband) Broadcast Channel with a single source s and multiple destinations $T = \{t_1, \dots, t_n\}$ (arranged in the order of increasing distance from s). From [5] and [6], we know that the superposition coding is equivalent to time sharing, which is optimal. Implying that the broadcast communication from a single source to multiple receivers can be decomposed into communication over n hyperarcs sharing the common source power. Therefore, we get the set of hyperarcs $\mathcal{A}_{bc} = \{(s, t_1), (s, t_1 t_2), \dots, (s, t_1 t_2 \dots t_n)\}$.

Similarly, in the Low-SNR (wideband) regime, interference becomes negligible with respect to noise, and all sources can achieve their point-to-point capacities analogous to Frequency Division Multiple Access (FDMA). In general, the MAC Channel consisting from n sources s_1, \dots, s_n transmitting to a common destination t can be interpreted as n point-to-point arcs each having point-to-point capacities. Thus, we get $\mathcal{A}_{mac} = \{(s_1, t), \dots, (s_n, t)\}$. Each hyperarc $(s, t_1 \dots t_j) \in \mathcal{A}_{bc} \cup \mathcal{A}_{mac}$ is associated with the rate function

$$R_{t_j}^s = \frac{P_{t_j}^s}{N_0 D_{st_j}^\alpha}, \forall j \in [1, n], \quad (13)$$

where $\alpha \geq 2$ is the path loss exponent.

B. Low-SNR Achievable Hypergraph Model

By concatenating the Low-SNR Broadcast Channel and MAC Channel models we obtain an Achievable Hypergraph Broadcast Model. For example the Broadcast Relay Channel consisting of a single source, n destinations and a relay. Although, the time sharing and FDMA are capacity achieving optimal schemes in the respective models, the Achievable Hypergraph Model is not necessarily capacity achieving. In contrast and more importantly for practical use, this model is easy to scale to larger and more complex networks.

The above Low-SNR Achievable Hypergraph Model also incorporates fading [1]. The rate function in Equation (13) is linear in transmitter power and convex in hyperarc distance, hence the results from this paper can be directly applied.

VI. CONCLUSION

We present simple and efficient geometry based algorithms for solving joint relay positioning and flow (max-flow/min-cost) optimization problems for a fairly general class of hypergraphs. Any application that satisfies the hypergraph construction rules and can be modeled under the classical multicommodity framework can use the results presented here.

As a part of future work it would be of interest to extend the work presented here to the general multicommodity setting where multiple sessions use a common relay.

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APPENDIX A
PROOF OF PROPOSITION 1

Proof: Let the set of nodes $\mathcal{N} = \{s, r, t_1, \dots, t_n\}$ be placed on the 2-D Euclidean plane and \mathcal{C} denote their convex hull polygon. Let us assume that the relay node r is placed outside the polygon \mathcal{C} , i.e. $z_r \notin \mathcal{C}$ and c be the nearest point to r in the polygon \mathcal{C} . Let the line segment joining z_r and c be denoted as $z_r - c$.

The rate over all the hyperarcs that either emanate from r or r is the farthest end node of the hyperarc, is relay position dependent. As the hyperarc rate is a decreasing function of distance, moving the relay closer to c on the segment $z_r - c$ decreases the distance between r and every point in the polygon \mathcal{C} and thus to every node in the system. This implies that for a given power allocation for the relay position dependent hyperarcs the rate can be increased as the relay gets closer to the point c . Consequently, we can conclude that the optimal relay position z_{\uparrow}^* maximizing the multicast flow F for the session (s, T) will lie in the convex hull polygon \mathcal{C} .

Similarly, all the relay position dependent hyperarcs will need lesser power to carry a given flow of value F as the relay r moves closer on the line segment $z_r - c$ to the point c . Implying, that for any target flow F the optimal relay position $z_{F\downarrow}^*$ will lie in \mathcal{C} . This concludes the proof. ■

APPENDIX B
PROOF OF THEOREM 1

Before we formally prove Theorem 1, we need to establish some basic tools from convex analysis.

Let $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be an increasing and convex function that maps a non-negative real input to a non-negative real output. Denote with $\bar{f}(x)$ the sub-derivative of $f(x)$ at the point $x \in \mathbf{R}$ and let $\partial f(x)$ denote the complete set of sub-derivatives at point x . If the set $\partial f(x)$ is a singleton set, then $\bar{f}(x) = \frac{\partial f}{\partial x}$, which is simply the derivative of the f at x ; else there exist a finite interval $\partial f(x)$ between the left and right limits of f at x . In addition, let us also assume that $f(0) \leq 0$. Then the following proposition is true.

Proposition 3: If f is any increasing convex function such that $f(0) \leq 0$ then

$$\sum_{i=1}^n f(x_i) \leq f\left(\sum_{i=1}^n x_i\right), \quad (x_1, \dots, x_n) \in \mathbf{R}^{n+}. \quad (14)$$

Proof: As f is increasing over the real line, for $x_1 < x_2$ we have $f(x_1) \leq f(x_2)$. Also, as f is convex $\bar{f}(x_1) \leq \bar{f}(x_2)$.

Let the slopes of line joining the points $(0, f(0))$ and $(x_1, f(x_1))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be given by,

$$s_1 = \frac{f(x_1) - f(0)}{x_1}, \quad s_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad (15)$$

where $0 < x_1 < x_2$ are points on real line. From the Generalized Mean Value Theorem we know that there always exist a point c and c' between $[0, x_1]$ and $[x_1, x_2]$, such that $\bar{f}(c) = s_1$ and $\bar{f}(c') = s_2$, respectively. This, along with the fact that f is increasing and convex implies,

$$\bar{f}(0) \leq \bar{f}(c) = s_1 \leq \bar{f}(x_1), \quad \bar{f}(x_1) \leq \bar{f}(c') = s_2 \leq \bar{f}(c').$$

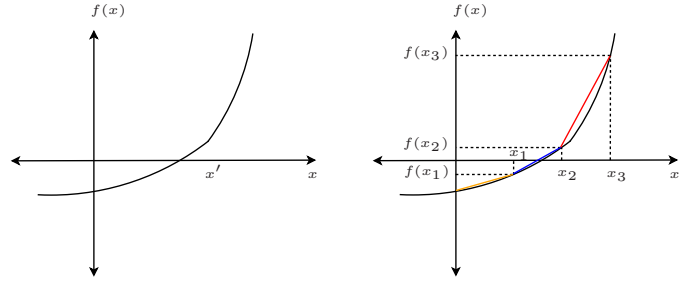


Fig. 5. (a): Increasing convex function f (possibly non-differentiable, e.g. at x'). (b): Orange, blue and red lines joining the points $(f(0), 0) - (f(x_1), x_1)$, $(f(x_1), x_1) - (f(x_2), x_2)$ and $(f(x_2), x_2) - (f(x_3), x_3)$, respectively.

In general, given n points $x_1 < \dots < x_n$ with s_i as the slope of line joining the points $(x_i, f(x_i)) - (x_{i+1}, f(x_{i+1}))$ we get,

$$s_1 \leq s_2 \leq \dots \leq s_{n-1}. \quad (16)$$

Consider now the four points $(0, f(0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$ and $(x_{12}, f(x_{12}))$, where $x_{12} = x_1 + x_2$. From Inequality 16 we get,

$$\frac{f(x_1) - f(0)}{x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_{12}) - f(x_2)}{x_2}. \quad (17)$$

Inequality 17 implies,

$$f(x_1) + f(x_2) - f(0) \leq f(x_{12}) = f(x_1 + x_2). \quad (18)$$

If $f(0) = 0$, then $f(x_1) + f(x_2) \leq f(x_1 + x_2)$. Using this fact it is straightforward to show that this also holds for $f(0) < 0$, for any $(x_1, x_2) \in \mathbf{R}^{2+}$.

Without loss of generality, assume that $x_{12} < x_3$, repeating the previous step of Inequalities 17-18 we get,

$$\begin{aligned} f(x_{12}) + f(x_3) &\leq f(x_{123}) \Rightarrow \\ f(x_1) + f(x_2) + f(x_3) &\leq f(x_1 + x_2 + x_3), \end{aligned}$$

where $x_{123} = x_1 + x_2 + x_3$. Similarly, repeating these n times we have

$$\sum_{i=1}^n f(x_i) \leq f\left(\sum_{i=1}^n x_i\right), \quad (x_1, \dots, x_n) \in \mathbf{R}^{n+}, \quad (19)$$

if $f(0) \leq 0$. This proves the proposition. ■

Now let f and g be increasing convex functions satisfying Proposition 3 and define $f_i(x_i) = \lambda_i f(x_i)$ and $g_i(y_i) = \lambda'_i g(y_i)$ as the $2n$ linear compositions of the function f and g for $i \in [1, n]$, where $\lambda_i \geq 0$, $\lambda'_i \geq 0$, $\forall i \in [1, n]$ and $f(0) = g(0) = 0$. Then consider the following program,

$$\begin{aligned} &\text{Maximize} \quad \left(\mathcal{F}_n = \sum_{i=1}^n \min(f_i(x_i), g_i(y_i)) \right) \quad (\text{T1A}) \\ &\text{subject to:} \quad \sum_{i=1}^n x_i \leq \mu, \quad \sum_{i=1}^n y_i \leq \nu, \\ &\quad \quad \quad x_i \geq 0, \quad y_i \geq 0, \quad \forall i \in [1, n]. \end{aligned}$$

Denote the set S^* as the set of optimizers of Program (T1A). In addition, assume that

$$\min(\lambda_k, \lambda'_k) = \max_{i \in [1, n]} (\min(\lambda_i, \lambda'_i)), \quad k \in [1, n]. \quad (20)$$

Let us denote a set of points $U^* = \{(\mathbf{x}_k, \mathbf{y}_k) = (0, \dots, 0, \mu, 0, \dots, 0), (0, \dots, 0, \nu, 0, \dots, 0)\}$ where, \mathbf{x}_k and \mathbf{y}_k are the vectors with μ and ν at the k^{th} place and all other elements 0, respectively. Then the following proposition is true.

Proposition 4: $U^* \subset S^*$.

Proof: Consider program (T1A) and without loss of generality assume that

$$\min(\lambda_1, \lambda'_1) \leq \min(\lambda_2, \lambda'_2) \leq \dots \leq \min(\lambda_n, \lambda'_n), \quad (21)$$

and

$$\min(\lambda_k, \lambda'_k) = \min(\lambda_{k+1}, \lambda'_{k+1}) = \dots = \min(\lambda_n, \lambda'_n), \quad (22)$$

where Equation (22) says that the last $n - k$ terms of Inequality (21) are equal. Let

$$\rho_i = \begin{cases} f_i(x_i) = \lambda_i f(x_i) & \text{if } \lambda_i \leq \lambda'_i, \\ g_i(y_i) = \lambda'_i g(y_i) & \text{if } \lambda_i > \lambda'_i. \end{cases}$$

Then, from Proposition 3 for any $0 \leq (\epsilon, \epsilon) \in \text{dom}(f, g)$ we get,

$$\sum_{i=1}^n \rho_i(\epsilon) \leq \rho_j(n\epsilon),$$

which implies

$$\sum_{i=1}^n \min(f_i(\epsilon), g_i(\epsilon)) \leq \min(f_j(n\epsilon), g_j(n\epsilon)), \quad (23)$$

as ρ_i is the limiting sub-term of the i^{th} term $\min(f_i(x_i), g_i(y_i))$ of function \mathcal{F}_n for any $i \in [1, n]$ and $j \in [k, n]$. From Inequality (23), we can infer that simply maximizing $\min(f_j(x_j), g_j(y_j))$ (for any $j \in [k, n]$) alone maximizes \mathcal{F}_n in Program (A), with all other terms attaining the value 0 (except for j^{th} term) because $\lambda_i f(0) = \lambda'_i f(0) = 0, \forall i \in [1, n]$. Therefore, $\mathcal{F}_n^* = \min(f_j(\mu), g_j(\nu))$ for any $j \in [k, n]$ is the maximum value of the function \mathcal{F}_n in Program (T1A).

Hence $U^* \subset S^*$, where $U^* = \{(\mathbf{x}_j, \mathbf{y}_j) = (0, \dots, 0, \mu, 0, \dots, 0), (0, \dots, 0, \nu, 0, \dots, 0) | j \in [k, n]\}$. ■

Under the same setting as for Program (T1A), consider the following program,

$$\text{Minimize} \quad \left(\mathcal{F}'_n = \sum_{i=1}^n (x_i + y_i) \right) \quad (T1B)$$

$$\text{subject to: } \zeta \leq \sum_{i=1}^n \min(f_i(x_i), g_i(y_i)),$$

$$\sum_{i=1}^n x_i \leq \mu, \quad \sum_{i=1}^n y_i \leq \nu,$$

$$x_i \geq 0, \quad y_i \geq 0, \quad \forall i \in [1, n],$$

where $\zeta \geq 0$ is a given positive real number and such that

$$\zeta \leq \min(f_i(\mu), g_i(\nu)), \quad \forall i \in [1, n]. \quad (24)$$

Denote with S'^* , the set of optimizers of Program (T1B). Then the following Proposition holds true.

Proposition 5: $(\mathbf{x}_{\mathbf{k}'}, \mathbf{y}_{\mathbf{k}'}) \in S'^*$.

Proof: Let us assume that the inverse functions f_i^{-1} and g_i^{-1} exists such that $f_i^{-1}(\lambda_i f(x_i)) = x_i$ in addition to $g_i^{-1}(\lambda'_i g(y_i)) = y_i$, for all $i \in [1, n]$. Also assume that for any $\epsilon \geq 0$ we get,

$$\begin{aligned} f_{k'}^{-1}(\lambda_{k'} f(x_{k'}^\epsilon)) + g^{-1}(\lambda_{k'} g(y_{k'}^\epsilon)) &= x_{k'}^\epsilon + y_{k'}^\epsilon \\ &= \min_{i \in [1, n]} (x_i^\epsilon + y_i^\epsilon), \end{aligned} \quad (25)$$

where $\lambda_i f(x_i^\epsilon) = \lambda'_i g(y_i^\epsilon) = \epsilon, \forall i \in [1, n]$. Since, f and g are increasing convex functions for non-negative input, their inverses f^{-1} and g^{-1} are increasing concave functions for non-negative input. From Proposition 3 we can deduce the reverse inequality for concave functions,

$$\sum_{i=1}^n f'(x_i) \geq f' \left(\sum_{i=1}^n x_i \right), \quad (26)$$

where $f' : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is an increasing concave function. From Inequality (26), we get for any $\epsilon \geq 0$

$$\begin{aligned} f^{-1}(\lambda_{k'} f(n x_{k'}^\epsilon)) + g^{-1}(\lambda_{k'} g(n y_{k'}^\epsilon)) \\ \leq \sum_{i=1}^n (f^{-1}(\lambda_i f(x_i^\epsilon)) + g^{-1}(\lambda'_i g(y_i^\epsilon))). \end{aligned} \quad (27)$$

Lastly, from Inequality (24) and (27) we conclude that $(\mathbf{x}_{\mathbf{k}'}, \mathbf{y}_{\mathbf{k}'}) \in S'^*$, where $\mathbf{x}_{\mathbf{k}'} = (0, \dots, 0, \zeta, 0, \dots, 0)$ and $\mathbf{y}_{\mathbf{k}'} = (0, \dots, 0, \zeta, 0, \dots, 0)$. ■

Note that we did not assume the differentiability of the functions f and g .

The set of paths L , that span all the destinations will be central to the proof of Theorem 1, so let us clarify some more notations. The path set $L = \{l_1, \dots, l_\tau\}$ contains only one path from s to T that does not go through r , namely $\{(s, t_1, \dots, r, \dots, t_n) = (s, T)\}$ and without loss of generality let us assume $l_\tau = (s, T)$. All other paths go through r and consist of two hyperarcs, i.e. $l_j = \{(s, T_1^j), (r, T_2^j)\}$ where $r \in T_1^j$ and $T_1^j \cup T_2^j = T, \forall j \in [1, \tau - 1]$. Then the path flow is given by

$$F_{l_j} = \begin{cases} \min(R_{T_1^j}^s, R_{T_2^j}^r) & = \min(\lambda_j^1 g(P_{T_1^j}^s), g(\lambda_j^2 P_{T_2^j}^r)) \\ & \text{if } j \in [1, \tau - 1], \\ R_T^s = \lambda_j g(P_T^s) & \text{if } j = \tau, \end{cases}$$

where $\lambda_j^1 = \frac{1}{h(D_{st_1^j})}$, $\lambda_j^2 = \frac{1}{h(D_{rt_2^j})}$, $\forall j \in [1, \tau - 1]$ and $\lambda_\tau = \frac{1}{h(D_{st_n})}$, where $t_1^j \in T$ and $t_2^j \in T$ are the farthest nodes from s and r spanned by the hyperarcs (s, T_1^j) and (r, T_2^j) , respectively.

Proof of Theorem 1: Consider the hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$ for the given position of relay $z_r \in \mathcal{C}$ and the path based formulation of multicast max-flow and min-cost problems in Program (A) and (B), respectively.

Since the hyperarcs are constructed in the order of increasing distance from the transmitter, there exist no two paths from the s to any destination $t_i \in T$ that are edge disjoint. This implies that only the paths spanning all the destinations in

the set L should to be considered, as sending any information over the paths that span a subset $T' \subset T$ has to be resent over at least one path in the set L that spans all the set T . This fact reduces Programs (A) and (B) to

Max-flow	Min-cost
Maximize $F = \sum_{l \in L} F_l$ (T1C) subject to: $F_i \leq \sum_{l \in L} F_l, \forall i \in [1, n]$, $F_l \geq 0, \forall l \in L$.	Minimize P (T1D) subject to: $F_i \leq \sum_{l \in L} F_l, \forall i \in [1, n]$, $F \leq F_i, \forall i \in [1, n]$, $F_l \geq 0, \forall l \in L$.

where $P = \sum_{j=1}^{\tau-1} (P_{T_1^{l_j}}^s + P_{T_2^{l_j}}^s) + P_T^s$ in Program (T1C), i.e. the sum of powers of all the hyperarcs of all the paths in L . Therefore,

$$F_i = \sum_{l \in L} F_l, \forall i \in [1, n], \implies F = \min_{i \in [1, n]} F_i = \sum_{l \in L} F_l.$$

Without loss of generality let us assume that among all the paths from s through r to T the path $l_k \in L$ has the highest min-cut, i.e. $\min(\lambda_{l_k}^1, \lambda_{l_k}^2) \geq \max_{j \in [1, \tau-1]} \lambda_{l_j}^1, \lambda_{l_j}^2$. Then we get two scenarios, either

$$\min(\lambda_1^1, \lambda_1^2) \leq \dots \leq \min(\lambda_k^1, \lambda_k^2) \leq \lambda_\tau, \quad (28)$$

where, the last inequality of Inequalities (28) says that the path (s, T) has the highest min-cut among all the paths in L . Then from Proposition 4, the multicast flow can be maximized by simply maximizing the flow over the path $l_\tau = (s, T)$, and since maximizing the flow over this path consumes all the source power μ the optimal multicast flow F^* is given by

$$F^* = F_{l_\tau}^* = R_{T_\tau}^* = \lambda_\tau g(\mu). \quad (29)$$

Otherwise if,

$$\min(\lambda_1^1, \lambda_1^2) \leq \dots \leq \lambda_\tau \leq \dots \leq \min(\lambda_k^1, \lambda_k^2), \quad (30)$$

then again by Proposition 4 maximizing the flow only over path l_k maximizes the multicast flow F in Program (T1C). Thus, we get

$$F_{l_k}^* = \min(R_{T_1^k}^{s*}, R_{T_2^k}^{r*}) = \min(\lambda_k^1 g(\mu), \lambda_k^2 g(\nu)). \quad (31)$$

Furthermore, if $\lambda_k^1 g(\mu) < \lambda_k^2 g(\nu)$, i.e. if the source has relatively more power than relay r , then the rest of the flow must be send over the path (s, T) as any other path through the relay (i.e. l_j where $j \neq k$ and $j \in [1, \tau-1]$) cannot be used due to no spare power left with relay. This implies

$$\begin{aligned} F^* &= \min(R_{T_1^k}^{s*}, R_{T_2^k}^{r*}) + \hat{R}_T^s \\ &= \min(\lambda_k^1 g(\mu'), \lambda_k^2 g(\nu)) + \lambda_\tau g(\mu - \mu'), \end{aligned} \quad (32)$$

where $\lambda_k^1 g(\mu') = \lambda_k^2 g(\nu)$ and $\hat{R}_T^s = R_T^s(P_T^s = \mu - \mu') = \lambda_\tau g(\mu - \mu')$. Thus, all the maximized multicast flow F^* goes over at most two paths, l_k and l_τ . Integrating Equations (31) and (32), we get

$$F^* = \begin{cases} R_T^{s*} & \text{if } \lambda_\tau = \max_{j \in [1, \tau]} (\min(\lambda_j^1, \lambda_j^2)), \\ \min(R_{T_1^k}^{s*}, R_{T_2^k}^{r*}) + \hat{R}_T^s & \text{if } \min(\lambda_k^1, \lambda_k^2) = \max_{j \in [1, \tau]} (\min(\lambda_j^1, \lambda_j^2)). \end{cases} \quad (33)$$

From Equation (33), we conclude that for any given position of relay $z_r \in \mathcal{C}$, the optimal multicast max-flow F^* goes over at most two paths namely l_k and l_τ . Consequently, this also holds true at $z_{\gamma\uparrow}^*$.

For the case of multicast min-cost Program (T1D) for the target flow $F \in [0, F^*]$, without loss of generality let us assume that

$$\lambda_{k'}^1 + \lambda_{k'}^2 = \max_{j \in [1, \tau-1]} (\lambda_j^1 + \lambda_j^2). \quad (34)$$

From Equation (5) cost of sending the flow $\epsilon > 0$ over the path $l_{k'}$ is given by

$$P_{T_1^{k'}, \epsilon}^s + P_{T_2^{k'}, \epsilon}^r = g^{-1}\left(\frac{\epsilon}{\lambda_{k'}^1}\right) + g^{-1}\left(\frac{\epsilon}{\lambda_{k'}^2}\right), \quad (35)$$

where g^{-1} is the inverse function of the power function g and is increasing and concave. From Equation (34) we get

$$P_{T_1^{k'}, \epsilon}^s + P_{T_2^{k'}, \epsilon}^r = \min_{j \in [1, \tau-1]} (P_{T_1^j, \epsilon}^s + P_{T_2^j, \epsilon}^r). \quad (36)$$

For a given position of relay $z_r \in \mathcal{C}$, then clearly $\min(\lambda_{k'}^1, \lambda_{k'}^2) = \min(\lambda_k^1, \lambda_k^2)$, i.e. the cheapest path through the relay is that path with the highest min-cut. This is true because

$$\begin{aligned} \min(\lambda_1^k, \lambda_2^k) &= \max_{j \in [1, \tau-1]} (\min(\lambda_1^j, \lambda_2^j)) \\ &= \frac{1}{\lambda_1^k} + \frac{1}{\lambda_2^k} = \min_{j \in [1, \tau-1]} \left(\frac{1}{\lambda_1^j} + \frac{1}{\lambda_2^j} \right), \end{aligned} \quad (37)$$

if $\lambda_2^k = \max_{j \in [1, \tau-1]} (\lambda_2^j)$ and this can be safely assumed for the path with the highest min-cut.

From Equation (36) and Proposition 5, we infer that $l_{k'}$ is the cheapest path for a unit flow among all the paths $l_j \in [1, \tau-1]$. Moreover from Equation (37), paths $l_{k'} (= l_k)$ and l_τ can carry any target multicast flow $F \in [0, F^*]$. So we get four cases

- (1) $P_{T_1^{k'}, \epsilon}^s + P_{T_2^{k'}, \epsilon}^r \leq P_{T_\tau, \epsilon}^s$ and $\min(\lambda_{k'}^1, \lambda_{k'}^2) \geq \lambda_\tau$,
- (2) $P_{T_1^{k'}, \epsilon}^s + P_{T_2^{k'}, \epsilon}^r \leq P_{T_\tau, \epsilon}^s$ and $\min(\lambda_{k'}^1, \lambda_{k'}^2) < \lambda_\tau$,
- (3) $P_{T_1^{k'}, \epsilon}^s + P_{T_2^{k'}, \epsilon}^r > P_{T_\tau, \epsilon}^s$ and $\min(\lambda_{k'}^1, \lambda_{k'}^2) \geq \lambda_\tau$,
- (4) $P_{T_1^{k'}, \epsilon}^s + P_{T_2^{k'}, \epsilon}^r > P_{T_\tau, \epsilon}^s$ and $\min(\lambda_{k'}^1, \lambda_{k'}^2) < \lambda_\tau$.

In all of the above cases, the target flow $F \in [0, F^*]$ flows over the paths $l_{k'}$ and l_τ only. Thus, we conclude that for any relay position $z_r \in \mathcal{C}$ the optimal min-cost target multicast flow F flows over at most two paths $l_{k'}$ and l_τ , and consequently also at the relay position $z_{F\downarrow}^*$. This completes the proof. ■

APPENDIX C PROOF OF PROPOSITION 2

Proof: From Theorem 1, we know that the maximized multicast flow goes over at most two paths, namely path \hat{l}_1 having the highest min-cut among the paths through r and path \hat{l}_2 that spans all the nodes in the system. Moreover, there always exist at least one relay position such that the min-cut of the path \hat{l}_1 is at least as that of path \hat{l}_2 , implying that this also

holds at the optimal relay position $z_{\gamma\uparrow}^*$ solving $(s, T, \mathcal{Z}, \gamma)$. This is also true for point p^* because at p^*

$$\max(h(D_{sp^*}), h(D_{p^*t_n})) \leq h(D_{st_n}),$$

that comes from its definition in Equation (8).

Positioning the relay at p^* will render the highest min-cut of path \hat{l}_1 compared to that for any other position. This is true from the definition of point p^* itself. If at point p^* we have $g(\nu)h(D_{sp^*}) = D_{p^*} = \min_{i \in [1, n]} g(\mu)h(D_{p^*t_i})$, then

$$F_{p^*}^* = \min\left(\frac{g(\mu)}{h(\pi_s^{p^*})}, \frac{g(\nu)}{h(\pi_r^{p^*})}\right),$$

where, $\pi_s^{p^*} = D_{sp^*}$ and $\pi_r^{p^*} = \max_{i \in [1, n]}(D_{p^*t_i})$.

From our assumption at point p^* the maximized flow F^* consumes all the source and the relay powers. Since we only consider those positions of relay at which the min-cut of path \hat{l}_1 is higher compared to path \hat{l}_2 , positioning the relay at any point $p \in \mathcal{C}$ such that $\pi_s^p > \pi_s^{p^*}$ only renders decreased maximum rate over the hyperarc $C_{T_1}^s$ of the path \hat{l}_1 . Implying that $F_p^* \leq F_{p^*}^*$, even though there might be some relay power left.

On the other hand, positioning the relay at point p such that $\pi_s^p < \pi_s^{p^*}$, increases the maximum rate over the hyperarc $C_{T_1}^s$, as

$$h(\pi_s^p) \leq h(\pi_s^{p^*}) \implies \frac{g(\mu)}{h(D_{sp})} \geq \frac{g(\mu)}{h(D_{sp^*})}.$$

Moreover, we get $\pi_r^p > \pi_r^{p^*}$, as moving away in any direction from point p^* increases $\max_{j \in T_2}(D_{p^*t_j})$. Therefore the multicast flow is at this point is given by,

$$F_p^* = \frac{g(\nu)}{h(\pi_r^p)} + \frac{g\left(\mu - g^{-1}\left(\frac{g(\nu)h(\pi_s^p)}{h(\pi_r^p)}\right)\right)}{h(D_{st_n})},$$

where the first term on the right hand side is the flow over the path $\hat{l}_1 = \{C_{T_1}^s, C_{T_2}^r\}$ that is limited the hyperarc $C_{T_2}^r$ and the second term is the flow over the path $\hat{l}_2 = \{C_{T_1}^s\}$ such that F_p^* is achieved by maximizing the flow over the paths \hat{l}_1 and \hat{l}_2 successively.

As a Corollary of Proposition 3, it can be seen that

$$a_1g(x_1) + a_2g(x_2) \leq a_3g(x_1 + x_2),$$

where g is an increasing convex function, $a_i \geq 0$ for $i \in [1, 3]$ are some constants such that $a_3 \geq \max(a_1, a_2)$. This implies that for any source power $\epsilon > 0$, the flow over the path \hat{l}_1 corresponding to the relay position p^* will always be larger than the sum flow over the paths \hat{l}_1 and \hat{l}_2 corresponding to the relay position p . Therefore, for any such relay position p , $F_p^* \leq F_{p^*}^*$. This proves the proposition. ■

APPENDIX D PROOF OF THEOREM 2

Refer Figure 2 as a reference example along with the proof.

Proof: Let the optimal relay positions $z_{\gamma\uparrow}^*$ and $z_{F\downarrow}^*$ be given that solve the problems $(s, T, \mathcal{Z}, \gamma)$ and $(s, T, \mathcal{Z}, \gamma, F)$,

respectively. Then the hyperarcs of the path $\hat{l}_1^\uparrow = \{C_{T_1}^s, C_{T_2}^r\}$ and $\hat{l}_1^\downarrow = \{C_{T_1}^s, C_{T_2}^r\}$ can be constructed simply by forming the first hyperarcs $C_{T_1}^s$ and $C_{T_1}^s$ with radii $\pi_s^{\uparrow*} = D_{sz_{\gamma\uparrow}^*}$ and $\pi_s^{\downarrow*} = D_{sz_{F\downarrow}^*}$, respectively. Here, the paths \hat{l}_1^\uparrow and \hat{l}_1^\downarrow represent the path \hat{l}_1 corresponding to the relay positions $z_{\gamma\uparrow}^*$ and $z_{F\downarrow}^*$, respectively. Compute the points

$$z_{r\uparrow} = \arg \max_{r \in C_{T_1}^s, \mathcal{Z} \in C_{\cup}} (\max(D_{rt})),$$

$$z_{r\downarrow} = \arg \max_{r \in C_{T_1}^s, \mathcal{Z} \in C_{\cup}} (\max(D_{rt})),$$

where $\hat{T}^\uparrow = \{t \in T | D_{st_j} > \pi_s^{\uparrow*} = D_{sz_{\gamma\uparrow}^*}\}$ and $\hat{T}^\downarrow = \{t \in T | D_{st} > \pi_s^{\downarrow*} = D_{sz_{F\downarrow}^*}\}$.

If the points r^\uparrow and r^\downarrow are not the same as $z_{\gamma\uparrow}^*$ and $z_{F\downarrow}^*$, respectively, then this contradicts the optimality of the two points $z_{\gamma\uparrow}^*$ and $z_{F\downarrow}^*$. This is true because the only way to either maximize the rate or minimize the cost over the hyperarcs $C_{T_1}^s$ and $C_{T_2}^r$ is to compute the points inside the hyperarcs $C_{T_1}^s$ and $C_{T_2}^r$ that minimize the maximum among the distances to all the destination nodes outside these hyperarcs from itself, respectively. Therefore, the optimal relay positions $z_{\gamma\uparrow}^*$ and $z_{F\downarrow}^*$ solving the problems $(s, T, \mathcal{Z}, \gamma)$ and $(s, T, \mathcal{Z}, \gamma, F)$, are the points of type \hat{r} on the curve $\hat{\mathcal{R}}$. Hence, $z_{F\downarrow}^* = z_{\gamma'\uparrow}^*$, for some $\gamma' \in (0, \infty)$.

Now let us consider that the position $z_{1\downarrow}^*$ that minimizes the cost of unit flow (normalized, if the flow values are less than unity) from s to T , and $z_{1\downarrow}^* = z_{\gamma\uparrow}^*$ for some $\hat{\gamma} \in (0, \infty)$. Without loss of generality, let us assume that $z_{1\downarrow}^*$ is situated on the right of $z_{\gamma\uparrow}^*$ on the segment $\hat{\mathcal{R}}$. This implies that the rate over the source hyperarc $R_{T_1}^{s, z_{\gamma\uparrow}^*}$ is the limiting term in

$$F_{\hat{l}_1^{z_{\gamma\uparrow}^*}}^* = \min(R_{T_1}^{s, z_{\gamma\uparrow}^*}, R_{T_2}^{r, z_{\gamma\uparrow}^*}),$$

where $F_{\hat{l}_1^{z_{\gamma\uparrow}^*}}^*$ is the maximized flow over the path $\hat{l}_1^{z_{\gamma\uparrow}^*}$ corresponding to the relay position $z_{\gamma\uparrow}^*$. Furthermore, the only way to increase the min-cut of the path \hat{l}_1 is to position the relay on the left of $z_{\gamma\uparrow}^*$ (closer to s and $z_{\gamma\uparrow}^*$) on the segment $\hat{\mathcal{R}}$, as positioning the relay further on the right of $z_{\gamma\uparrow}^*$ on the segment $\hat{\mathcal{R}}$ will not only increase the cost of unit flow but will also decrease the min-cut of the path \hat{l}_1 . Therefore, for any $F \in [0, F^*]$, $z_{F\downarrow}^* \in z_{\gamma\uparrow}^* - z_{\gamma\uparrow}^*$, where $z_{\gamma\uparrow}^* - z_{\gamma\uparrow}^*$ is the sub-segment of the piecewise linear segment $\hat{\mathcal{R}}$ joining the point $z_{\gamma\uparrow}^*$ and $z_{\gamma\uparrow}^*$. The same argument holds for the case when the point $z_{\gamma\uparrow}^*$ lies on the right of $z_{\gamma\uparrow}^*$. Thus, we can conclude that

$$z_{F\downarrow}^* = z_{\gamma\uparrow}^*,$$

for some $\hat{\gamma} \in [\min(\bar{\gamma}, \gamma), \max(\bar{\gamma}, \gamma)]$, and $\forall F \in [0, F^*]$.

Now suppose that for some $\hat{\gamma} \in \gamma - \bar{\gamma}$ such that $\hat{\gamma} = \frac{\nu'}{\mu'}$ and $(\mu', \nu') \preceq (\mu, \nu)$, the optimal relay position maximizing the multicast flow F is given by $z_{\gamma\uparrow}^* \in z_{\gamma\uparrow}^* - z_{\gamma\uparrow}^*$ and the maximized multicast flow is given by $F_{z_{\gamma\uparrow}^*}^*$. Implying that, we

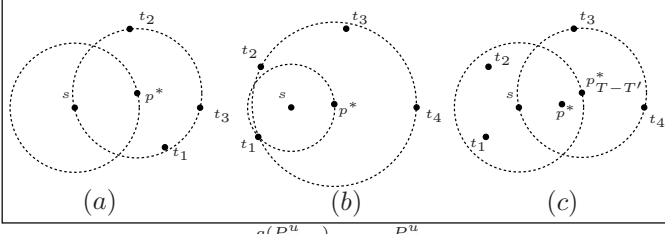


Fig. 6. Consider $R_{V_{ku}}^u = \frac{g(P_{ku}^u)}{h(D_{uvku})} = \frac{P_{ku}^u}{(D_{uvku})^2}$ with $g(\mu) = g(\nu) = 1$ in the examples. (a): $|T| = 3$ example with $\frac{1}{(D_{sp^*})^2} = \frac{1}{(D_{p^*t_n})^2}$ and the path $\hat{l}_1 = \{C_{T1}^s, C_{T2}^s\}$ (corresponding to p^*) carries all the maximized flow F^* . Therefore, $z_{\gamma\uparrow}^* = z_{p^*}$. (b): $|T| = 4$ example showing the path $\hat{l}_1 = \{C_{T1}^s, C_{T2}^s\}$ corresponding to p^* but $\frac{1}{(D_{sp^*})^2} > \frac{1}{(D_{p^*t_4})^2}$. (c): The nodes in $T' = \{t_1, t_2\}$ can be ignored, as the hyperarc C_{T1}^s of the path $\hat{l}_1 = \{C_{T1}^s, C_{T2}^s\}$ corresponding to $p_{T\setminus T'}^*$ spans the set T' . Thus $z_{\gamma\uparrow}^* = z_{p_{T\setminus T'}^*}$.

need at least the total source and relay power of value $\mu' + \nu'$ to achieve the multicast flow $F_{z_{\gamma\uparrow}^*}^*$. Thus, we can conclude that $z_{\gamma\uparrow}^* = z_{F_{z_{\gamma\uparrow}^*}^*}^*$. Hence, for any $\gamma \in \gamma - \overline{\gamma}$, there exist a flow value $F_{z_{\gamma\uparrow}^*}^* \in [0, F^*]$ such that $z_{\gamma\uparrow}^* = z_{F_{z_{\gamma\uparrow}^*}^*}^*$. This completes the proof of Theorem 2. ■

APPENDIX E

PROOF OF OPTIMALITY OF MAX-FLOW ALGORITHM

We divide the space of max-flow problem in three categories, each for a step in the Max-flow Algorithm in Figure 3. Proving the optimality for each category will prove the optimality of the Max-flow Algorithm as any given problem instance $(s, T, \mathcal{Z}, \gamma)$ will fall in one of the three categories. The first two categories (corresponding to the first two steps of the Max-flow Algorithm) deal with the instances when at the optimal relay position $z_{\gamma\uparrow}^*$ all the maximized multicast flow F^* goes over the single path \hat{l}_1 .

Proof of Step 1: Refer Appendix C for the proof of Proposition 2 and Figure 6(a) for an example. ■

Proof of Step 2: Now assume that $g(\nu)h(D_{sp^*}) < g(\mu)h(D_{p^*t_n})$. This implies that there exist at least one destination node that is closer to s than to p^* , thus we can build the ordered set $T' = \{t'_j \in T | D_{st'_j} < D_{p^*t'_j}\} = \{t'_1, \dots, t'_j\}$ in increasing distance from s .

We can then recompute the point p^* for the destination nodes in the set $T \setminus T'$ and denote it by $p_{T \setminus T'}^*$. If we get, $g(\nu)h(D_{sp_{T \setminus T'}^*}) = g(\mu)h(D_{p_{T \setminus T'}^*t_n})$, using Proposition 2 we infer that the point $p_{T \setminus T'}^*$ is the optimal relay position maximizing the multicast flow from s to the set of destinations in the set $T \setminus T'$. Furthermore, if $D_{st'_j} \leq D_{sp_{T \setminus T'}^*}$ then all the nodes in the set $T \setminus T'$ are spanned by the hyperarc C_{T1}^s with radii $\pi_s = D_{sp_{T \setminus T'}^*}$ corresponding to the relay position $p_{T \setminus T'}^*$. Implying that the $z_{\gamma\uparrow}^* = z_{p_{T \setminus T'}^*}$. ■

Step 2 essentially gets rid of redundant destinations for computing the point p^* that are close enough to the source s . Refer Figure 6 for an example.

In contrast, if at point $p_{T \setminus T'}^*$ we get $g(\nu)h(D_{sp_{T \setminus T'}^*}) < g(\mu)h(D_{p_{T \setminus T'}^*t_n})$, then we can divide the problem of optimal

relay position within two regions in \mathcal{C} . Since the relay position maximizing the multicast flow in the two regions is going to be unique, we can compare the two results and declare the global optimal position $z_{\gamma\uparrow}^*$ solving the problem instance $(s, T, \mathcal{Z}, \gamma)$.

First region is the interior of the circle $\text{int}C_{t'_j}^s$ centered at s with radius $D_{st'_j}$, and the second region as the area $\mathcal{C} \setminus \text{int}C_{t'_j}^s$ which is the rest of region in \mathcal{C} minus the first region. Let us denote the optimal relay positions inside the first and second regions with z_1^* and z_2^* , respectively. At first, we present the following two propositions that will come in handy for the proof of optimality of Step 3. Refer Figure 7 with $g(P) = P$ and $h(D) = D^2$ for an example.

Proposition 6: For any $|T| = 2$ case max-flow problem instance such that $g(\nu)h(D_{sp^*}) < \max_{i \in [1,2]}(g(\mu)h(D_{p^*t_i})) = D_{p^*}$, $z_1^* \in \text{int}C_{t_1}^s$ lies on the line segment p^* and p_{12} .

Proof: Consider the two destination node case such that $g(\nu)h(D_{sp^*}) < \max_{i \in [1,2]}(g(\mu)h(D_{p^*t_i}))$, where

$$p^* = \arg \min_{p \in \mathcal{C}} (\max(g(\nu)h(D_{sp}), g(\mu)h(D_{pt_1}), g(\mu)h(D_{pt_2}))).$$

This implies that $D_{st'_j} > D_{sp_{T \setminus T'}^*}$, where

$$p_{T \setminus T'}^* = \arg \min_{p \in \mathcal{C}} (\max(g(\nu)h(D_{sp}), g(\mu)h(D_{pt_2}))),$$

and $T' = \{t_1\}$. Moreover, as a consequence of Theorem 1, only those points need to be considered such that positioning the relay at this point gives higher min-cut for the path \hat{l}_1 than that of path \hat{l}_2 , inside the region $\mathcal{C} \cap \text{int}C_{t_1}^s$.

The circle $C_{t'_j}^s$ is the circle $C_{t_1}^s$ centered at s with radius D_{st_1} . Let the perpendicular bisector of the nodes t_1 and t_2 be denoted as \perp_{12} , and its intersection point with the segment $s - t_2$ by p_{12} . Now consider any point $p \in \mathcal{C} \cap \text{int}C_{t_1}^s$ in the halfplane containing the node t_2 of the bisector \perp_{12} and let p' denote the closest point on \perp_{12} to p . Then clearly, $D_{sp'} < D_{sp} \Rightarrow h(D_{sp'}) \leq h(D_{sp})$ and $D_{p't_1} = D_{p't_2} < D_{pt_1} \Rightarrow h(D_{p't_2}) \leq h(D_{pt_1})$. This implies that the min-cut of the path \hat{l}_1 reduces when the relay is positioned at the point p compared to that at point p' . Therefore, for any relay position in the halfplane (of \perp_{12}) containing t_2 , there always exist a position on the segment $p^* - p_{12}$ that is a better candidate for the optimal relay position maximizing the multicast flow.

Now consider a point $p \in \mathcal{C} \cap \text{int}C_{t_1}^s$ in the halfplane (of \perp_{12}) containing the node t_1 such that $D_{sp} < D_{sp'}$, then we get $D_{pt_2} > D_{p't_2}$. Note that, positions p for which $D_{sp} > D_{sp'}$ are uninteresting due to the fact that not only the min-cut of the path \hat{l}_1 decreases but also that the maximum possible rate over the hyperarc C_{T1}^s decreases compared to the relay position p' (corresponding to p). Then the maximized multicast flow for the relay position p and p' such that $D_{sp} < D_{sp'}$ is given by

$$F_p^* = \frac{g(\nu)}{h(D_{pt_2})} + \frac{g\left(\mu - g^{-1}\left(\frac{g(\nu)h(D_{sp})}{h(D_{pt_2})}\right)\right)}{h(D_{st_2})},$$

$$F_{p'}^* = \frac{g(\nu)}{h(D_{p't_2})} + \frac{g\left(\mu - g^{-1}\left(\frac{g(\nu)h(D_{sp'})}{h(D_{p't_2})}\right)\right)}{h(D_{st_2})},$$

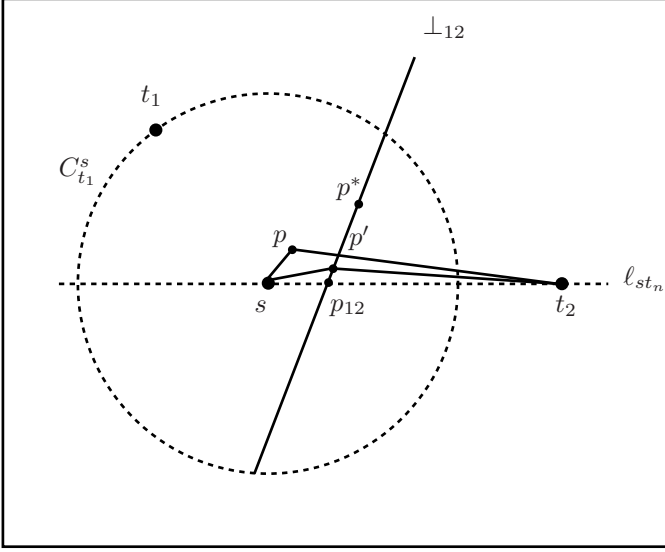


Fig. 7. $|T| = 2$ node example with $T' = \{t_1\}$. Note that, $D_{sp} < D_{sp'}$ and $D_{pt_2} < D_{p't_2}$. Therefore, the rate over the paths $\hat{l}_1^p = \{(s, p), (p, t_1t_2)\}$ and $\hat{l}_1^{p'} = \{(s, p'), (p', t_1t_2)\}$ is limited by the hyperarcs (p, t_1t_2) and (p', t_1t_2) , respectively.

respectively. The first term in the above equations is the flow $F_{\hat{l}_1} = \min(R_{T_1}^s, R_{T_2}^r) = R_{T_2}^r$ over the path \hat{l}_1 which is limited by the rate $R_{T_2}^r$ over the hyperarc $C_{T_2}^r$ and the second term is the flow over the path \hat{l}_2 which is the function of total source power μ minus the power used over the hyperarc $C_{T_1}^s$. As $D_{p't_2} < D_{pt_2}$, we have $\frac{g(\nu)}{h(D_{p't_2})} \leq \frac{g(\nu)}{h(D_{pt_2})}$. Furthermore, from triangle inequality we get $D_{sp'} - D_{sp} < D_{p't_2} - D_{pt_2}$, implying that

$$\frac{g(\epsilon)}{h(D_{sp})} - \frac{g(\epsilon)}{h(D_{sp'})} \leq \frac{g(\epsilon)}{h(D_{p't_2})} - \frac{g(\epsilon)}{h(D_{pt_2})},$$

for any $P_{T_1}^s = P_{T_2}^r = \epsilon \geq 0$. Therefore, we get more flow over the path \hat{l}_1 corresponding to the relay position p' compared to that of relay position p . Lastly, since the min-cut of the path \hat{l}_2 is strictly smaller than that of path \hat{l}_1 , it can be concluded that $F_p^* \leq F_{p'}^*$. Hence, for any point $p \in \mathcal{C} \cap \text{int}C_{t_1}^s$ there exist a point p' on the segment $p^* - p_{12}$ of the perpendicular bisector \perp_{12} , such that $F_p^* \leq F_{p'}^*$, hence $z_1^* \in p^* - p_{12}$. ■

Note that if h is an strictly increasing function of distance, then the following inequality would hold strictly $F_p^* < F_{p'}^*$. For $|T| > 2$, Proposition 6 can be generalized in the following way. For simplicity, let us assume that the line ℓ_{st_n} passing through s and t_n is horizontal and point p^* lies above ℓ_{st_n} (ref. Figure 7 for example). Now, let the set of perpendicular bisectors for each pair of nodes in T be denoted by $\perp = \{\perp_{12}, \perp_{13}, \dots, \perp_{n-1n}\}$, where \perp_{ab} denotes the perpendicular bisector of the nodes $t_a \in T$ and $t_b \in T$, and $|\perp| = \frac{n!}{n!(n-2)!}$. Most of the bisectors $\perp_{ab} \in \perp$ will intersect with the line ℓ_{st_n} and let \angle_{ab} denote the angle between the point s , the point of intersection of \perp_{ab} and ℓ_{st_n} , and any point on the bisector \perp_{ab} above the line ℓ_{st_n} . The closest point on the perpendicular bisector \perp_{ab} to s is denoted by p_{ab}^s . In addition, let $\perp \supset \bar{\perp} = \{\perp_{\bar{1}n}, \perp_{\bar{2}n}, \dots, \perp_{\bar{m}n}\}$ be the subset of bisectors $\perp_{\bar{j}n}$ of the nodes $\bar{t}_j \in T$ and t_n (the farthest node from s

in the system) for $j \in [1, m]$, such that there exist a segment $p_{\bar{j}n} - q_{\bar{j}n}$ of $\perp_{\bar{j}n}$ in \mathcal{C} so that the farthest nodes from any point $p \in p_{\bar{j}n} - q_{\bar{j}n}$ are the nodes \bar{t}_j and t_n themselves. For example, in Figure 7 the segment $p^* - p_{12}$ of bisector \perp_{12} . Finally, without loss of generality we assume that $\perp_{\bar{1}n} \in \bar{\perp}$ be the perpendicular bisector such that the point $p_{\bar{1}n}^s$ lies on the segment $p_{\bar{1}n} - q_{\bar{1}n}$.

Proposition 7: For $|T| = n > 2$ max-flow problem instance such that $g(\nu)h(D_{sp^*}) < D_{p^*} = \max_{i \in [1, n]}(g(\mu)h(D_{p^*t_i}))$, $z_1^* \in \mathcal{C} \cap \text{int}C_{t_{j'}}^s$ lies on the piecewise linear segment $(p^* - q_{\bar{1}n}, q_{\bar{1}n} - q_{\bar{2}n}, \dots, q_{\bar{l-1}n} - q_{\bar{l}n})$.

Proof: Consider the region $\mathcal{C} \cap \text{int}C_{t_{j'}}^s$ for determining the best possible relay position z_1^* maximizing the multicast flow F . p^* is already a good reference point in $\mathcal{C} \cap \text{int}C_{t_{j'}}^s$. From Proposition 6, we know that position p such that $D_{sp} < D_{sp^*}$, i.e. the rate over the hyperarc $C_{T_1}^s$ increases for a given power $P_{T_1}^s$, is an interesting position in terms of finding the point z_1^* , thus we will only consider such directions from p^* . In other words, only those bisectors $\perp_{\bar{j}n} \in \bar{\perp}$ need to be considered that intersect ℓ_{st_n} on the segment $s - t_n$ and make an obtuse angle $\angle_{\bar{j}n}$.

Let $t_{\bar{1}}$ be the limiting node in determining p^* , i.e. $g(\mu)h(D_{p^*t_{\bar{1}}}) = D_{p^*}$ whose bisector makes the largest obtuse angle $\angle_{\bar{1}n}$ with ℓ_{st_n} (e.g. node t_1 in Figure 8(a)). Also, $t_{\bar{1}}$ and t_n are the farthest limiting nodes from s in determining the point p^* . This implies, that there exist a segment $p^* - q_{\bar{1}n}$ on the bisector $\perp_{\bar{1}n}$ (towards s) such that $t_{\bar{1}}$ and t_n are the farthest nodes from any point $p \in p^* - q_{\bar{1}n}$. If $p_{\bar{1}n}^s \in p^* - q_{\bar{1}n}$, then using Proposition 6 for any position $p \in \mathcal{C} \cap C_{t_{j'}}^s$, and the closest point $p'_{\bar{1}n}$ to p on $\perp_{\bar{1}n}$, we get three cases

$$\begin{aligned} \text{either } D_{sp} &< D_{sp'_{\bar{1}n}} \text{ and } D_{pt_n} > D_{p'_{\bar{1}n}t_n}, \\ \text{or } D_{sp} &> D_{sp'_{\bar{1}n}} \text{ and } D_{pt_n} > D_{p'_{\bar{1}n}t_n}, \\ \text{or } D_{sp} &> D_{sp'_{\bar{1}n}} \text{ and } D_{pt_n} < D_{p'_{\bar{1}n}t_n}. \end{aligned}$$

Then if $p_{\bar{1}n}^s \in p_{\bar{1}n} - q_{\bar{1}n}$, using Proposition 6 we can deduce that $z_{\gamma\uparrow}^* \in p^* - q_{\bar{1}n}$. All other points $p \in \mathcal{C} \cap \text{int}C_{t_{j'}}^s$ such that $p'_{\bar{1}n}$ lies outside the segment $p^* - q_{\bar{1}n}$ can be ignored.

On the other hand, suppose that $p_{\bar{1}n}^s \notin p_{\bar{1}n} - q_{\bar{1}n}$. Then, there exist another bisector $\perp_{\bar{2}n}$ (of the nodes $t_{\bar{2}}$ and t_n) that intersects $\perp_{\bar{1}n}$, say at point $q_{\bar{1}n}$ and contains a segment $q_{\bar{1}n} - q_{\bar{2}n} \in \mathcal{C} \cap \text{int}C_{t_{j'}}^s$, such that for any point $p \in q_{\bar{1}n} - q_{\bar{2}n}$, $t_{\bar{2}}$ and t_n are the farthest nodes in the system. Using proposition 6 again, we can infer that for all the points $p \in \mathcal{C} \cap \text{int}C_{t_{j'}}^s$ such that $D_{pp'_{\bar{1}n}} \leq D_{pp'_{\bar{2}n}}$, positioning the relay at $p'_{\bar{1}n}$ will render $F_p^* \leq F_{p'_{\bar{1}n}}^*$. Similarly, if $D_{pp'_{\bar{1}n}} > D_{pp'_{\bar{2}n}}$, then positioning the relay at $p'_{\bar{2}n}$ will render $F_p^* \leq F_{p'_{\bar{2}n}}^*$. Finally, if $p_{\bar{2}n}^s \in q_{\bar{1}n} - q_{\bar{2}n}$, then we can conclude that $z_{\gamma\uparrow}^* \in (p_{\bar{1}n} - q_{\bar{1}n}, q_{\bar{1}n} - q_{\bar{2}n})$.

Generalizing to case of l such bisectors such that $p_{\bar{l}n}^s \in q_{\bar{l-1}n} - q_{\bar{l}n}$, we conclude that $z_1^* \in (p_{\bar{1}n} - q_{\bar{1}n}, q_{\bar{1}n} - q_{\bar{2}n}, \dots, q_{\bar{l-1}n} - q_{\bar{l}n})$. The same argument would suffice if p^* lies below the line ℓ_{st_n} and this completes the proof. ■

Now we are ready to prove the optimality of Step 3 of the Max-flow Algorithm.

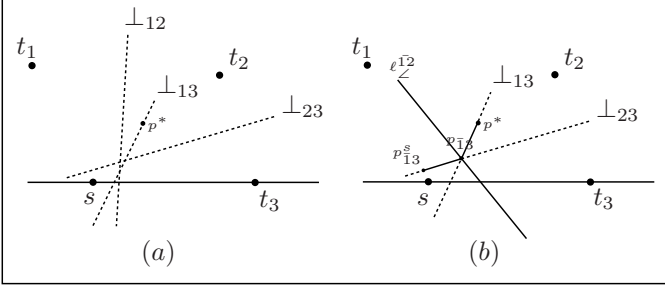


Fig. 8. $|T| = 3$ example with $T' = \{t_1\}$. (a): $\perp = \{\perp_{12}, \perp_{13}, \perp_{23}\}$. (b): $\perp = \{\perp_{13}, \perp_{23}\}$, where $t_1 = t_1$ and $t_2 = t_2$. ℓ_{12}^{12} is the angle bisector of $\angle p_{13}^* p_{23}^*$ dividing the region $\mathcal{C} \cap C_{t_1}^s$ into the two halfplanes of points that are closer to segments $p_{13} - q_{13}$ and $q_{13} - q_{23}$.

Proof of Step 3: Following Step 2, assume that not all the nodes in the set T' lie inside the hyperarc $C_{T_1}^s$ with radius $\pi_s = D_{sp_{T-T'}}^*$, then by reforming the hyperarc $C_{T_1}^s$ with radius $\pi_s = D_{st_{j'}}^*$, we can compute the point

$$\hat{p} = \arg \min_{p \in C_{T_1}^s} \left(\max_{t_j \in T \setminus \{T'\}} (D_{pt_j}) \right). \quad (38)$$

The point \hat{p} will always lie on the circumference of the hyperarc circle $C_{T_1}^s$. Positioning the relay r at \hat{p} gives the hyperarc $C_{T_2}^r$ with radius $\pi_r = D_{\hat{p}t_n}$, thus rendering the path $\hat{l}_1 = \{C_{T_1}^s, C_{T_2}^r\}$ for the relay position \hat{p} . Since $D_{sp_{T-T'}}^* < D_{st_{j'}}^* = D_{s\hat{p}}$, this implies $g(\nu)h(D_{s\hat{p}}) \geq g(\mu)h(D_{\hat{p}t_n})$. Thus maximizing the rate over the path \hat{l}_1 will alone maximize the multicast flow F for the position \hat{p} , giving

$$F_{\hat{p}}^* = \min \left(\frac{g(\mu)}{h(D_{s\hat{p}})}, \frac{g(\nu)}{h(D_{\hat{p}t_n})} \right). \quad (39)$$

For any relay position p with $z_p \in \mathcal{C} \setminus \text{int} C_{T_1}^s$ the min-cut of the path \hat{l}_1 reduces compared to the position \hat{p} simply because $D_{s\hat{p}} < D_{sp}$, implying that the maximized multicast rate $F_p^* \leq F_{\hat{p}}^*$. Therefore, $z_2^* = z_{\hat{p}}^*$.

For all the positions inside the circle $C_{t_{j'}}^s$, from Proposition 7, we know that $z_1^* \in (p^* - q_{1n}, q_{1n} - q_{2n}, \dots, q_{l-1n} - q_{ln})$, where $(p^* - q_{1n}, q_{1n} - q_{2n}, \dots, q_{l-1n} - q_{ln})$ is piecewise linear segment made up of the sub-segments of perpendicular bisectors $\perp = \{\perp_{1n}, \dots, \perp_{ln}\}$, where from any point on the bisector \perp_{kn} , t_k and t_n are the farthest nodes in the system, for all $k \in [1, l]$. Let us re-denote this piecewise linear segment by $s = (s_{1n}, \dots, s_{ln})$, where s_{kn} denotes the sub-segment $q_{k-1n} - q_{kn}$ for $k \in [1, n]$.

It is easy to compute the equation of each bisector in \perp . Let the equations be given by

$$y = m_{\bar{1}}x + c_{\bar{1}}, \quad \forall k \in [1, l], \quad (40)$$

where $y = m_{\bar{1}}x + c_{\bar{1}}$ is the equation of the bisector \perp_{kn} and the segment $q_{k-1n} - q_{kn}$ can be represented by limiting the values of $x \in [x_{q_{k-1n}}, x_{q_{kn}}]$, where $z_{q_{k-1n}} = (x_{q_{k-1n}}, y_{q_{k-1n}})$ are the coordinates of the point q_{k-1n} in the plane. Moreover, as we know that for any point $p_{\bar{k}} \in q_{k-1n} - q_{kn}$, the limiting hyperarc is $C_{T_2}^r$, implying that the maximized multicast flow $F_{p_{\bar{k}}}^*$ can be achieved simply by maximizing the flows over the

paths \hat{l}_1 and \hat{l}_2 in succession, we get

$$F_{p_{\bar{k}}}^* = \frac{g(\nu)}{h(D_{p_{\bar{k}}t_k})} + \frac{g\left(\mu - g^{-1}\left(\frac{h(D_{sp_{\bar{k}}})g(\nu)}{h(D_{p_{\bar{k}}t_k})}\right)\right)}{h(D_{st_n})},$$

where the only variables are $D_{sp_{\bar{k}}}$ and $D_{p_{\bar{k}}t_k}$ as $p_{\bar{k}} \in q_{k-1n} - q_{kn}$. The variables $D_{sp_{\bar{k}}}$ and $D_{p_{\bar{k}}t_k}$ further are functions of coordinates $(x_{p_{\bar{k}}}, y_{p_{\bar{k}}})$ of point $p_{\bar{k}}$. Using the Equation (40), we can rewrite $F_{p_{\bar{k}}}^* = F_{\bar{k}}^*(x_{\bar{k}})$ as a function of single variable $x_{\bar{k}}$, where $x_{\bar{k}} \in [x_{q_{k-1n}}, x_{q_{kn}}]$. Then the optimal relay position maximizing the multicast flow F in the region $\mathcal{C} \cap \text{int} C_{t_{j'}}^s$ is given by $z_1^* = (x_1^*, y_1^*)$, where

$$x_1^* = \arg \max_{x_{\bar{k}}} \left(\max_{\bar{k} \in [\bar{1}, \bar{l}]} F_{\bar{k}}^*(x_{\bar{k}}) \right), \quad (41)$$

where $x_{\bar{k}} \in [x_{q_{k-1n}}, x_{q_{kn}}]$, $\forall \bar{k} \in [\bar{1}, \bar{l}]$ and $y_1^* = m_{\bar{k}}x_{\bar{k}}^* + c_{\bar{k}}$.

Finally, comparing the values of F_1^* and F_2^* , the optimal relay position solving $(s, T, \mathcal{Z}, \gamma)$ is given by

$$z_{\gamma}^* = \begin{cases} z_1^* & \text{if } F_1^* > F_2^* \\ z_2^* & \text{if } F_1^* < F_2^* \end{cases}$$

This completes the proof of optimality of Step 3 of Max-flow Algorithm. ■

Remark on solving Equation (41): The single variable function $F_{\bar{k}}^*(x_{\bar{k}})$ is non-convex and smooth over the domain $x_{\bar{k}} \in [x_{q_{k-1n}}, x_{q_{kn}}]$, $\forall \bar{k} \in [\bar{1}, \bar{l}]$. Moreover it can be proven that there exist a single stationary point (maxima) of the function in the domain $[x_{q_{k-1n}}, x_{q_{kn}}]$. Therefore, using gradient based approach the global maxima can be achieved, implying that non-convexity is not a hinderance.

APPENDIX F

PROOF OF OPTIMALITY OF MIN-COST ALGORITHM

Assume that for a given problem instance $(s, T, \mathcal{Z}, \gamma, F)$ the Inequality (12) holds and at the optimal relay position $z_{F\downarrow}^*$ all the target flow F goes over path \hat{l}_1 only. Refer Figures 9-10 for example.

Proof of Step 2: Assume that at the optimal relay position, all the min-cost multicast flow will go over path \hat{l}_2 only. This implies that the intersection region $C'_{\cap} = C'^s_{\cap} \cap C'^{t_n}$ of the circles C'^s and C'^r with radii $\pi'_s = h^{-1}\left(\frac{g(\mu)}{F}\right)$ and $\pi'_{t_n} = h^{-1}\left(\frac{g(\nu)}{F}\right)$, respectively, contains at least one point, and this assures feasibility. Positioning the relay at any point inside C'_{\cap} will fetch the multicast flow of value F over the path \hat{l}_2 . Implying that the relay position in C'_{\cap} minimizing the cost of unit flow also minimizes the cost of flow F and is thus the optimal relay position solving $(s, T, \mathcal{Z}, \gamma, F)$.

Now consider the point

$$\hat{p} = \arg \min_{p \in C'_{\cap}} \left(h(D_{sp}) + \max_{i \in [1, n]} (h(D_{pt_i})) \right). \quad (42)$$

It can be seen that all the nodes in the set $\hat{T} = \{\hat{t} \in T | D_{s\hat{t}} \leq D_{s\hat{p}}\}$ can be ignored as they are close enough to the source

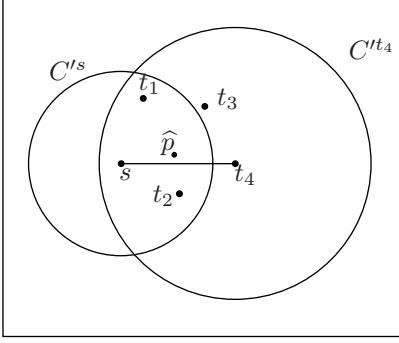


Fig. 9. Consider the $|T| = 4$ node system, with $C'_\cap = C'^{ts} \cap C'^{t4}$ and the point \hat{p} that is the optimal relay position minimizing the cost of unit flow over the path \hat{l}_1 in the region $C'_\cap \cap C'^{ts}$.

node to be spanned by the source hyperarc $C'^{ts}_{T_1}$ of the path \hat{l}_1 . Therefore, we can recompute

$$\hat{p} = \arg \min_{p \in C'_\cap} (h(D_{sp}) + \max_{t \in T \setminus \{\hat{T}\}} (h(D_{pt_i}))),$$

and get rid of unnecessary bias. Denote the cost of unit flow $\Psi_{\hat{p}} = h(D_{s\hat{p}}) + h(D_{\hat{p}t_n})$ corresponding to the relay position \hat{p} .

Now consider the region $C'_\cap \cap C'^{ts}_{\hat{p}}$, where the circle $C'^{ts}_{\hat{p}}$ has the radius $D_{s\hat{p}}$. \hat{p} is the optimal relay position in the region minimizing the cost of flow F , as all the destination nodes inside the circle $C'^{ts}_{\hat{p}}$ are not the limiting nodes determining the position \hat{p} . But the point \hat{p} is not necessarily the global optimum for the whole region C'_\cap . So we break the problem of finding the global optimal relay position minimizing the cost of multicast flow F , into finding the optimal relay position among disjoint regions of C'_\cap and then compare them to declare the global optimal relay position.

Denote the set of nodes $\bar{T} = \{t \in T \setminus \{\hat{T}, t_n\} | D_{st} > \pi_{\hat{p}}^{\bar{p}}, D_{st} \leq \pi_s^{\bar{p}}\} = \{\bar{t}_1, \dots, \bar{t}_l\}$, where $\pi_{\hat{p}}^{\bar{p}} = D_{s\hat{p}}$. Consider finding the optimal relay position \bar{p}_1 in the region $C'_\cap \cap (\bar{C}_2^s \setminus \bar{C}_1^s)$ that minimizes the cost of unit flow over the path \hat{l}_1 , where circles \bar{C}_1^s and \bar{C}_2^s are centered at s with radii $D_{s\bar{t}_1}$ and $D_{s\bar{t}_2}$, respectively. Then the problem can be stated as,

$$\hat{p}_1 = \arg \min_{p \in \bar{C}_1^s} (\max(h(D_{sp}), h(D_{s\bar{t}_1})) + \max_{t \in \bar{T}_1} (h(D_{pt}))), \quad (43)$$

where the set $\bar{T}_1 = \{t \in T | D_{st} > D_{s\bar{t}_1}\}$ consists of destination nodes lying outside the circle \bar{C}_1^s . The Program in Equation (43) outputs for the optimal relay position minimizing the cost of unit flow over the hyperarc \hat{l}_1 for the relay position in the region $C'_\cap \cap (\bar{C}_2^s \setminus \bar{C}_1^s)$. Although, the region $C'_\cap \cap (\bar{C}_2^s \setminus \bar{C}_1^s)$ is non-convex, the optimization program in Equation (43) is convex and easy to solve. Similarly, for all the nodes in the set $\bar{t}_j \in \bar{T}$, we can compute

$$\hat{p}_j = \arg \min_{p \in \bar{C}_j^s} (\max(h(D_{sp}), h(D_{s\bar{t}_{j-1}})) + \max_{t \in \bar{T}_j} (h(D_{pt}))), \quad (44)$$

and the cost of unit flow at the optimal relay positions \hat{p}_j in the disjoint region $C'_\cap \cap (\bar{C}_{j+1}^s \setminus \bar{C}_j^s)$ can be calculated, thus denote the vector $\bar{\Psi} = \{\bar{\Psi}_1, \dots, \bar{\Psi}_l\}$. Furthermore, generating the set $\bar{\Psi}$ needs $< |T| = n$ iterations. Finally, we get

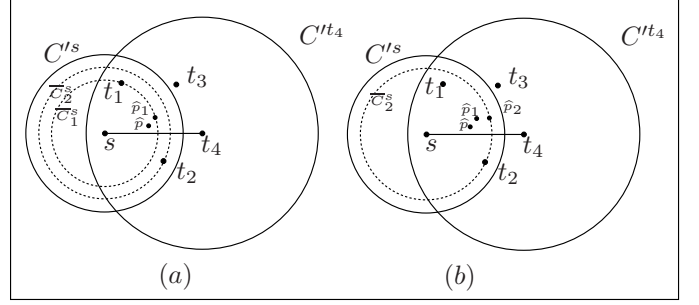


Fig. 10. $|T| = 4$ example of Figure 9 is considered with $\bar{T} = \{t_1, t_2\}$. (a): \hat{p}_1 on the circumference of dotted circle \bar{C}_1^s is shown, which is the optimal relay position solving $(s, T, \mathcal{Z}, \gamma, F)$ in the region $\bar{C}_2^s \setminus \bar{C}_1^s$. (b) For the region $C'^{ts} \setminus \bar{C}_2^s$ and the optimal point \hat{p}_2 is shown on the circumference of \bar{C}_2^s .

$$z_{F\downarrow}^* = \begin{cases} z_{\hat{p}} & \text{if } \Psi_{\hat{p}} \leq \bar{\Psi}_m, \\ z_{\bar{p}_m} & \text{if } \Psi_{\hat{p}} \geq \bar{\Psi}_m, \end{cases}$$

where, $\bar{\Psi}_m = \min_{j \in [1, l]} (\bar{\Psi}_j)$.

Note that, the dividing of the region C'_\cap into the disjoint non-convex regions $C'_\cap \setminus \text{int} \bar{C}_j^s$ lets capture the idea of positioning the relay anywhere in the region C'_\cap such that the source hyperarc $C'^{ts}_{T_1}$ of the path \hat{l}_1 spans all the destination nodes that are closer to source than relay and this ensures the global optimality. This completes the Proof. \blacksquare